An axiomatization of real numbers in constructive type theory and its application

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Online

Motivation and Overview

- certified computations over real numbers from constructive proofs
- axiomatize real number computation in constructive type theory
- properties of real number computation have to be reflected in the axiomatization

  partiality and nondeterminism
- propose a sound axiomatization of partial and nondeterministic computation, and real numbers
- implement the axiomatization

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Talk overview:

1. Intro: constructive type theory + why axiomatization of reals not trivial
2. Axiomatization of partiality, nondeterminism, and reals
3. Some examples
4. Implementation and evaluation results
5. Conclusion

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The belief in the universal validity of the principle of the excluded third in mathematics is considered by the intuitionists as a phenomenon of the history of civilization of the same kind as the former belief in the rationality of π, or in the rotation of the firmament about the earth - L. E. J. Brouwer

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- We need a framework where "proof"s become objects

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- Proofs $\simeq$ λ-terms having transparent computational content

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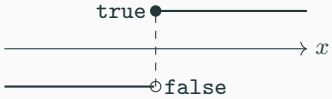
- Classical axiomatization of reals R is invalid:

$$\forall(x : \mathsf{R}).\ x \geq 0 \vee x < 0$$

The sign function (any discontinuous function) is uncomputable



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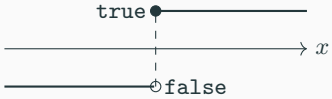

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  but cannot do any effective reasoning

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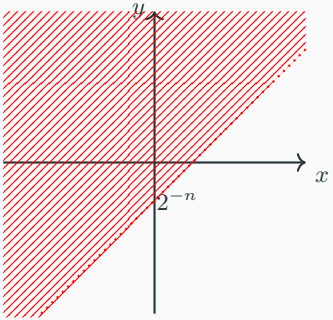

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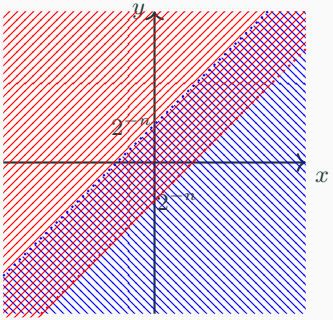

Nondeterministic COMPUTATION

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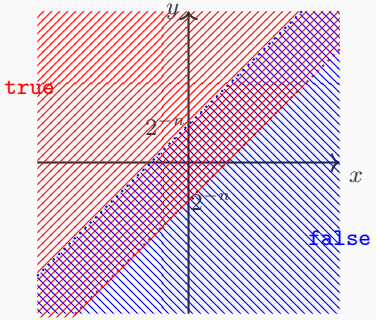

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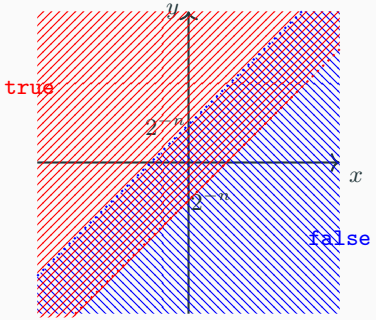

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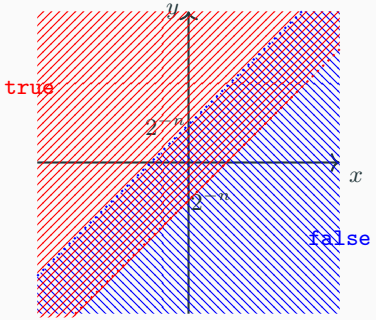

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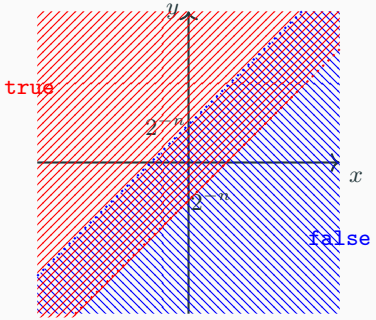

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- Partiality and Nondeterminism need to be expressible in logical counterpart
- Expressive enough for

$$\forall (x, y : \mathsf{R}).\ x < y \quad \textit{“is semi-decidable”}$$

- Expressive enough for

$$\forall (x, y : \mathsf{R}).\ x < y + 2^{-n} \quad \textit{“nondeterministically OR”}\ y < x + 2^{-n}$$

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Example

- $\forall(n, m : \mathsf{N}).\ \mathsf{semi}(n = m)$
- Define $\mathsf{dec}(P) :\equiv P \vee \neg P$. Then, $\mathsf{dec}(P) \to \mathsf{semi}(P)$
- $\mathsf{semi}(P) \to \mathsf{semi}(\neg P) \to \mathsf{dec}(P)$?

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- There is a moand $(\mathsf{M} : \mathsf{Type} \to \mathsf{Type}, \eta^{\mathsf{M}}, \mu^{\mathsf{M}}, \mathsf{lift}^{\mathsf{M}})$
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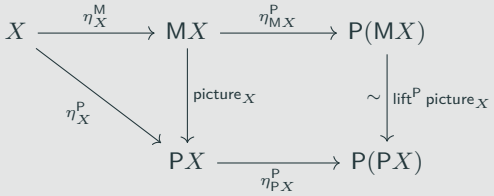

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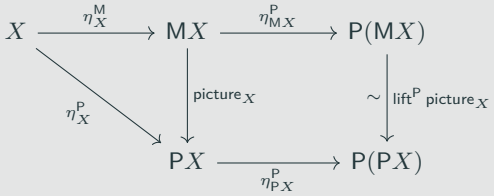

- and admits subsingleton elimination

$$\big(\forall(x, y : X).\ x = y\big) \to (X \simeq \mathsf{M}X)$$

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- Let $A \mathbin{\mathring{\vee}} B$ be the nondeterministic type $\mathsf{M}\big(A \vee B\big)$

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- Suppose $f : A \to \exists(x : C).\ P(x)$
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- If $A \mathbin{\mathring{\vee}} B$ is known, we have $\mathring{\exists}(x : C).\ P(x)$

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Definition

semi $P := \exists(s : \mathsf{K}).\ P \leftrightarrow s \downarrow$

Lemma

choose : $\forall(P, Q : \mathsf{Prop}).\ \mathit{semi}(P) \rightarrow \mathit{semi}(Q) \rightarrow P\ \tilde{\vee}\ Q \rightarrow\ \ P\ \mathring{\vee}\ Q$

Example (Dijkstra style guarded expressions)

$$\mathsf{case}\ p\ q\ r\ s:$$
$$\mathsf{semi}(p) \to \mathsf{semi}(q) \to p \mathbin{\tilde{\vee}} q \to (p \to r) \to (q \to s) \to (p \wedge r) \mathbin{\mathring{\vee}} (q \wedge s)$$

Proof

Qed

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- choose yields $p \mathbin{\mathring{\vee}} q$

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Proof

- choose yields $p \mathbin{\mathring{\vee}} q$
- We can construct a term $t : p \vee q \to (p \wedge r) \vee (q \wedge s)$
- We can lift it to get $p \mathbin{\mathring{\vee}} q \to (p \wedge r) \mathbin{\mathring{\vee}} (q \wedge s)$

Qed

Nondeterministic Types: Examples

Example

$\forall(P : \mathsf{Prop}).\ \mathsf{semi}(P) \to \mathsf{semi}(\neg P) \to \mathsf{dec}(P)$

Proof

Qed

Nondeterministic Types: Examples

Example

$\forall(P : \mathsf{Prop}).\ \mathsf{semi}(P) \to \mathsf{semi}(\neg P) \to \mathsf{dec}(P)$

Proof

- $\mathsf{LEM}(P) : P\ \tilde{\vee}\ \neg P$

Qed

Nondeterministic Types: Examples

Example

$\forall(P : \mathsf{Prop}).\ \mathsf{semi}(P) \to \mathsf{semi}(\neg P) \to \mathsf{dec}(P)$

Proof

- $\mathsf{LEM}(P) : P \mathbin{\tilde{\vee}} \neg P$
- Let $p_1 : \mathsf{semi}(P)$ and $p_2 : \mathsf{semi}(\neg P)$

Qed

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Proof

- $\mathsf{LEM}(P) : P \mathbin{\tilde{\vee}} \neg P$
- Let $p_1 : \mathsf{semi}(P)$ and $p_2 : \mathsf{semi}(\neg P)$
- choose $P\ \neg P\ p_1\ p_2\ (\mathsf{LEM}(P)) : P \mathbin{\mathring{\vee}} \neg P$

Qed

Example

$\forall(P : \mathsf{Prop}).\ \mathsf{semi}(P) \to \mathsf{semi}(\neg P) \to \mathsf{dec}(P)$

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- $P \vee \neg P$ is subsingleton

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- choose $P\ \neg P\ p_1\ p_2\ (\mathsf{LEM}(P)) : P \mathbin{\mathring{\vee}} \neg P$
- $P \vee \neg P$ is subsingleton
- extraction axiom yields $P \vee \neg P$

Qed

Axiom (Reals $\mathbb{R}$)

- R is a type with two known elements $0, 1 : \mathsf{R}$.
- $+, -, \times, ^{-1}, >$ are term constants
- The axioms are inhabited

	field axioms	A_1	$0 \neq 1$
A_2	$x + y = y + x$	A_3	$x + y + z = x + (y + z)$
A_4	$x + 0 = x$	A_5	$x - x = 0$
A_6	$x \times y = y \times x$	A_7	$x \times y \times z = x \times (y \times z)$
A_8	$x \times 1 = x$	A_9	$\forall (t : x \neq 0).\ x \times t^{-1} = 1$
A_{10}	$x \times (y + z) = x \times y + x \times z$		order axioms
A_{11}	$x = y \ \tilde{\vee}\ x < y \ \tilde{\vee}\ y < x$	A_{12}	$x < y \to y < z \to x < z$
A_{13}	$x < y \to \neg y < x$	A_{14}	$x < y \to x + z < y + z$
A_{15}	$0 < z \to x < y \to z \times x < z \times y$	A_{16}	$\mathsf{semi}(x < y)$
	metric completeness	A_{17}	"something about limit"

1. Intro: constructive type theory + why axiomatization of reals not trivial
2. Axiomatization of partiality, nondeterminism, and reals
3. **Some examples**
4. Implementation and evaluation results
5. Conclusion

Example

SoftComp : $\forall(x, y, \varepsilon : \mathsf{R}).\ 0 < \varepsilon \to (y - \varepsilon < x \mathbin{\mathring{\vee}} x - \varepsilon < y)$

Proof

Qed

Example

SoftComp : $\forall(x, y, \varepsilon : \mathsf{R}).\ 0 < \varepsilon \to (y - \varepsilon < x \mathbin{\mathring{\vee}} x - \varepsilon < y)$

Proof

- $t : y - \varepsilon < x \mathbin{\tilde{\vee}} x - \varepsilon < y$ form weak total order

Qed

Example

SoftComp : $\forall(x, y, \varepsilon : \mathsf{R}).\ 0 < \varepsilon \to (y - \varepsilon < x \mathbin{\mathring{\vee}} x - \varepsilon < y)$

Proof

- $t : y - \varepsilon < x \mathbin{\tilde{\vee}} x - \varepsilon < y$ form weak total order
- $a : \mathsf{semi}(y - \varepsilon < x)$ and $b : \mathsf{semi}(y - \varepsilon < x)$ are from axioms

Qed

Example

SoftComp : $\forall(x, y, \varepsilon : \mathsf{R}).\ 0 < \varepsilon \to (y - \varepsilon < x \mathbin{\mathring{\vee}} x - \varepsilon < y)$

Proof

- $t : y - \varepsilon < x \mathbin{\tilde{\vee}} x - \varepsilon < y$ form weak total order
- $a : \mathsf{semi}(y - \varepsilon < x)$ and $b : \mathsf{semi}(y - \varepsilon < x)$ are from axioms
- choose $a\ b\ t$ is a proof for $(y - \varepsilon < x \mathbin{\mathring{\vee}} x - \varepsilon < y)$

Qed

Safe order comparison

Example

$\forall(x : \mathsf{R})\ x \neq 0 \to 0 < x \vee x < 0$

Proof

Qed

Example

$\forall(x : \mathsf{R})\ x \neq 0 \to 0 < x \vee x < 0$

Proof

- From weak total order, $0 < 0\ \tilde{\vee}\ 0 < x$

Qed

Example

$\forall(x : \mathsf{R})\ x \neq 0 \to 0 < x \vee x < 0$

Proof

- From weak total order, $0 < 0 \,\tilde{\vee}\, 0 < x$
- From choose and semi, we have $0 < x \,\mathring{\vee}\, x < 0$

Qed

Safe order comparison

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- $0 < x \vee x < 0$ sub-singleton

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Safe order comparison

Example

$\forall(x : \mathsf{R})\ x \neq 0 \to 0 < x \vee x < 0$

Proof

- From weak total order, $0 < 0 \,\tilde{\vee}\, 0 < x$
- From choose and semi, we have $0 < x \,\mathring{\vee}\, x < 0$
- $0 < x \vee x < 0$ sub-singleton
- Extraction axiom yields $0 < x \vee x < 0$

Qed

Advanced Examples

- Metric completeness axiomatized as "every effective Cauchy sequence constructive has limit"
- Mixing it with the nondeterminism, can prove
  - construction of maximum

$$\forall(x, y : \mathsf{R}).\ \exists(z : \mathsf{R}).\ x < y \to z = y \wedge (\neg x < y) \to z = x$$

  - construction of square root

$$\forall(x : \mathsf{R}).\ 0 \leq x \to \exists(y : \mathsf{R}).\ y \times y = x$$

  - constructive Intermediate Value Theorem

$$\forall(f : \mathsf{R} \to \mathsf{R}).\ f(0) < 0 < f(1) \to (\tilde{\exists}!(x : \mathsf{R}).\ f(x) = 0) \to \exists!(x : \mathsf{R}).\ f(x) = 0)$$

    assuming the continuity principle

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Program Extraction

When we prove a statement

$$\forall(x : \mathsf{R}).\ \exists(y : \mathsf{R}).\ P\ x\ y$$

we get a computable function $f : \mathbb{R} \to \mathbb{R}$ computing numbers with the property P.

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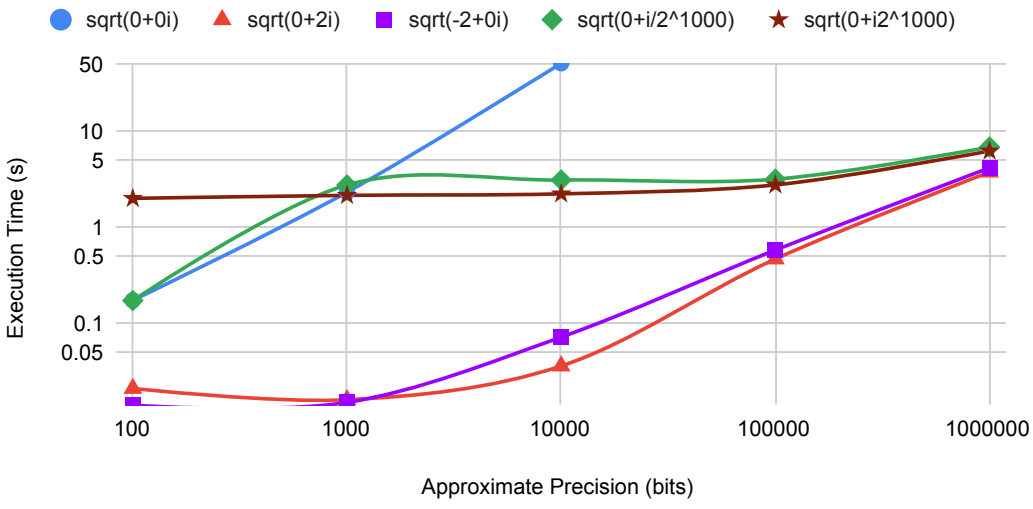
we get a multivalued function $f : \mathbb{R} \rightrightarrows \mathbb{R}$.
In the Coq implemtation this is realized by mapping R to AERN's datatype `CReal` and axiomatized operations to primitive operations in AERN.

Quality of programs: Execution speed

Benchmark		Average execution time (s)			
Formula	Accuracy	Extracted	Hand-written	Native	iRRAM
$\max(0, \pi - \pi)$	10^6 bits	3.5	3.8	3.8	1.59
$\sqrt{2}$	10^6 bits	0.72	0.70	0.40	0.62
$\sqrt{\sqrt{2}}$	10^6 bits	1.52	1.38	0.85	1.15
$x - 0.5 = 0$	10^3 bits	1.44	0.32	—	0.03
$x(2 - x) - 0.5 = 0$	10^3 bits	2.02	0.35	—	0.04
$\sqrt{x + 0.5} - 1 = 0$	10^3 bits	12.9	2.35	—	0.29

(i7-4710MQ CPU, 16GB RAM, Ubuntu 18.04, Haskell Stackage LTS 17.2)

Quality of programs: Execution speed



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Summary

- We specified a set of axioms for real numbers and nondeterministic computation over them in a constructive type theory
- proved various - simple but interesting to advanced - examples showing the usefulness of this axiomatization
- (omitted from the talk) proved the soundness by extending the realizability interpretation of the type theory
- (omitted from the talk) extended the nondeterminism monad with nondeterministic choice principle
- (omitted from the talk) proved the nondeterministic existence of complex square roots
- implemented them in Coq and extracted to AERN/Haskel programs

Available in detail in:


Michal Konečný, Sewon Park, and Holger Thies.
Axiomatic Reals and Certified Efficient Exact Real Computation.
WoLLIC 2021. Springer, Cham, 2021.


Michal Konečný, Sewon Park, and Holger Thies.
Extracting efficient exact real number computation from proofs in constructive type theory.
Arxiv preprint (arXiv:2202.00891).